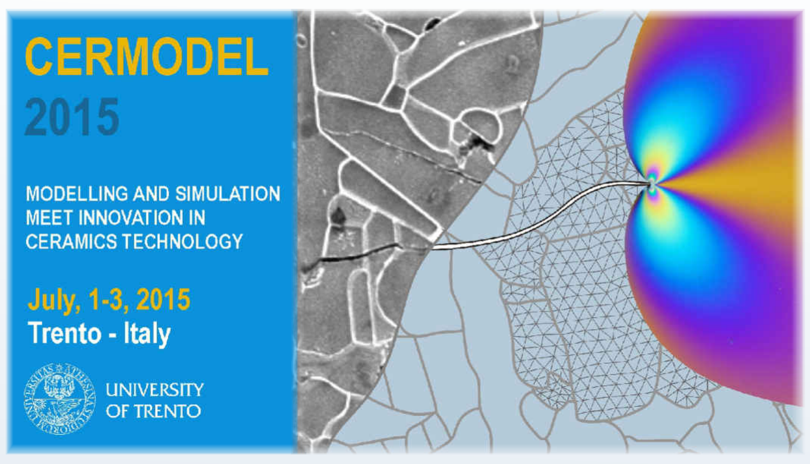

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MODELS FOR DYNAMICAL INSTABILITIES INDUCED BY COULOMB FRICTION

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INTRODUCTION

The behaviour of an elastic system in stability problems strongly depends on the type of force that loads the structure. Statically the compressive critical load wherewith a beam becomes instabile is given by the Euler formula

$$P_{cr} = \left(\frac{\pi}{l_0}\right)^2 EJ$$

When a structure is subjected to a follower forces, the statical approach does not show all the possible instabilities that the system can exhibit: in these cases the way to determinate the critical loads is via a dynamical approach.

Classical examples are represented by the Ziegler's column (Fig. 1), in the setting of discrete systems and the Beck's column, in the setting of continuous systems (Fig. 2). Both the structures are subjected to a compressive load that follows the deformed shape during the growth of the motion. Even systems loaded with a tensile load, for instance two rigid bars connected through a slider and externally constrained by a hinge (Fig. 3), show dynamical instabilities.

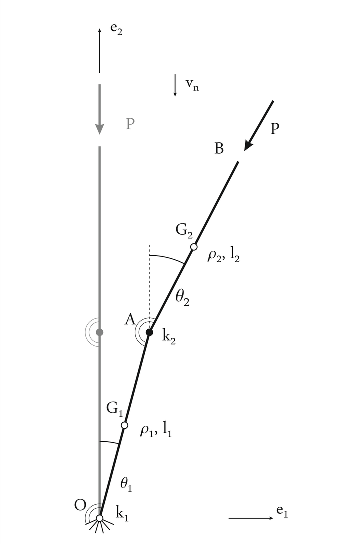

Figure 1: The Ziegler's column, a two-degree-of-freedom system subjected to a compressive follower load.

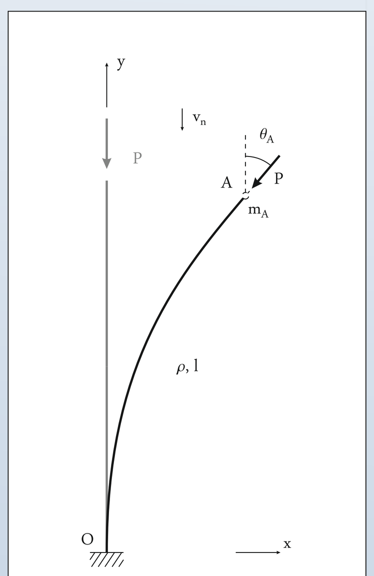

Figure 2: The Beck's column, a continuous system subjected to a compressive follower load.

ZIEGLER'S COLUMN

The system[1] described in Figure 1 is composed by two rigid rods (with mass per unit length ρ_1 and ρ_2) where the rotational springs of stiffness k_1 and k_2 provide the elasticity. The defomed shape of the structure is fully determined by the two Lagrangian parameters θ_1 and θ_2. The follower load is induced to the column with a wheel sliding with Coulomb friction in the plane that moves with velocity v_n.

The two nonlinear differential equations of motion for the system, assuming that the hinges are viscoelastic of the Voigt type with constants β_1, β_2, k_1 and k_2, is found with the virtual work principle

$$\begin{cases} \frac{1}{3}\rho l_1^2(l_1+3l_2)\ddot{\theta}_1 + \frac{1}{2}\rho l_1 l_2^2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + \frac{1}{2}\rho l_1 l_2^2 \sin(\theta_1-\theta_2)\dot{\theta}_2^{\,2} + \\ + (k_1+k_2)\theta_1 - k_2\theta_2 + (\beta_1+\beta_2)\dot{\theta}_1 - \beta_2\dot{\theta}_2 - P(\dot{B}_p^r)\sin(\theta_1-\theta_2)l_1 = 0 \\ \frac{1}{2}\rho l_1 l_2^2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + \frac{1}{3}\rho l_2^3\ddot{\theta}_2 - \frac{1}{2}\rho l_1 l_2^2\sin(\theta_1-\theta_2)\dot{\theta}_1^{\,2} + \\ - k_2(\theta_1-\theta_2) - \beta_2(\dot{\theta}_1-\dot{\theta}_2) = 0 \end{cases}$$

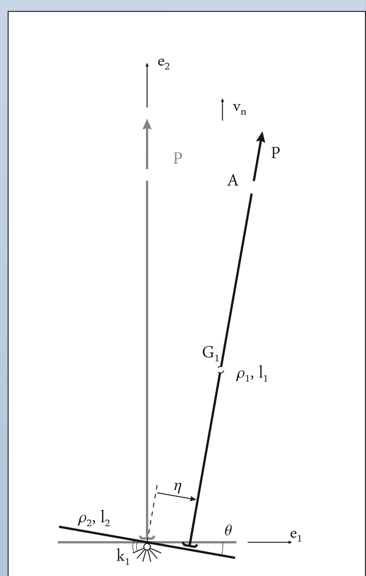

Figure 3: A two-degree-of-freedom discrete system subjected to a tensile follower load.

The numerical solution for the problem encounters difficulties solving the multivalue and discontinuous relation of the friction law, therefore in this work we will use the simplification employed by Oden and Martins

$$\mu\left(\dot{B}_p^r\right) = \mu_d \begin{cases} \text{sign}\left(\dot{B}_p^r\right), & \text{se } \dot{B}_p^r \notin [-\varepsilon,\varepsilon] \\ \left(2-\frac{|\dot{B}_p^r|}{\varepsilon}\right)\frac{\dot{B}_p^r}{\varepsilon}, & \text{se } \dot{B}_p^r \in [-\varepsilon,\varepsilon] \end{cases}$$

where ε is a small parameter.

If we consider the absence of viscosity (β_1=β_2=0) the solutions of the linearized system can be written in the matrix form

$$\begin{pmatrix} (1+\lambda/3)\omega^2+\gamma-k-1 & \omega^2/(2\lambda)-\gamma+1 \\ \omega^2/(2\lambda)+1 & \omega^2/(3\lambda^2)-1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where A_j are the (complex) amplitudes, Ω is the circular frequency and the other parameters are

$$\gamma = \frac{Pl_1}{k_2} \qquad \omega^2 = \frac{\rho l_1^2 l_2}{k_2}\Omega^2 \qquad k = \frac{k_1}{k_2} \qquad \lambda = \frac{l_1}{l_2}$$

Non-trivial solution to the system is possible if the determinant of the matrix becomes zero. In that case three possibilities arise:

- two real positive values for ω^2, a situation that corresponds to **stability** of the system ($\gamma<\gamma_f$). It is shown in Figure 3;
- two complex conjugate values for ω^2, a situation where the oscillation of the system becomes steady at large deformation ($\gamma_f<\gamma<\gamma_d$). This instability is called **flutter**. It is shown in Figure 4;
- two real and negative values for ω^2, a situation corresponding to vibrations that grow exponentially with time ($\gamma>\gamma_d$). This instability is called **divergence**.

The critical load for flutter and divergence instability is written in the adimensional form

$$\gamma_{f,d} = \frac{k+(1+\lambda)^3 \mp \lambda\sqrt{k(3+4\lambda)}}{1+(3/2)\lambda}$$

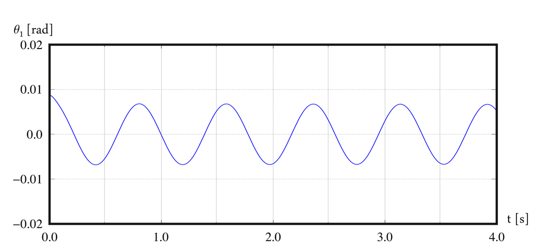

Figure 4: Parameter θ_1 with a compressive load P=3.4 N and v_n=0.3 m/s. The system is stable.

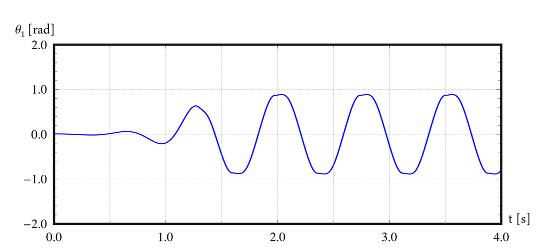

Figure 5: Parameter θ_1 with a compressive load P=6.8 N and v_n=0.3 m/s. The system shows flutter instability.

BECK'S COLUMN

The stability of a continuous beam subjected to a compressive follower force (with mass per unit length ρ and constant flexural rigidity EJ) was studied for the first time by Beck[2]. He considers an elastic cantilever subjected to the load P at the end tip. The system described in Figure 2 differs from the aforementioned rod for a light concentrated mass m_A at the free end. The load cell that measures the load applied to the system, the ball bearing that provide the frictional force and the ball rollers that transmit the load to cantilever constitute the tip mass.

The equation of elastica that governs the problem can be written as

$$EJy''''(x,t) + Py''(x,t) + \rho\ddot{y}(x,t) = 0$$

with the following boundary condition

$$y(0,t) = y'(0,t) = 0, \qquad y''(l,t) = y'''(l,t) = 0$$

Rewriting the problem, adopting a separation of variables for the y(x,t)=f(x)g(t), the function f(x) is expressed by the general solution

$$\begin{aligned} f(x) &= \textstyle\sum_{j=1}^4 C_j \exp(i\lambda_j x) = \\ &= C_1 \sinh\lambda_1 x + C_2\cosh\lambda_1 x + C_3\sin\lambda_2 x + C_4\cos\lambda_2 \end{aligned}$$

so that the roots of the characteristic equation result

$$\lambda_{1,2} = \left[\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 + a\omega^2}\right]^{1/2} \qquad \text{with} \qquad \frac{P}{EJ} = p, \quad \frac{\rho}{EJ} = a$$

Inserting the boundary conditions in the function f(x) and imposing zero the determinant of the homogeneous system we find the characteristic equation that relates the natural frequency ω of the system with the adimensional applied force p

$$g(p,\omega^2) = (2a\omega^2+p^2) + 2a\omega^2\cosh(\lambda_1 l)\cos(\lambda_2 l) + p\sqrt{a\omega^2}\sinh(\lambda_1 l)\sin(\lambda_2 l)$$

The lower load for which the system becomes instabile and shows flutter is calculated imposing that the derivative of g(p, ω^2) respect of the natural frequency ω is zero, so that the critical load is written as

$$pl^2 = 20.051 \longrightarrow P_{cr} = 2.031\left(\frac{\pi}{l}\right)^2$$

Notice that for the Beck's column the divergence instability does not occur like in Ziegler'column.

The numerical analisys of the problem (Figure 6) has been performed with the software ABAQUS, considering a polycarbonate beam of length l=0.3 m with density ρ=1240 kg/m^3, Young's modulus E=2250 MPa and Poisson ratio v=0.37. The point mass at the free end weigh 47 g, the cantilever is loaded with a compressive follower force of 37.6 N and the velocity of the plane is 0.3 m/s. The Martins condition for the definition of the friction factor μ_d is implemented in the input file via subroutine UAMP written in Fortran.

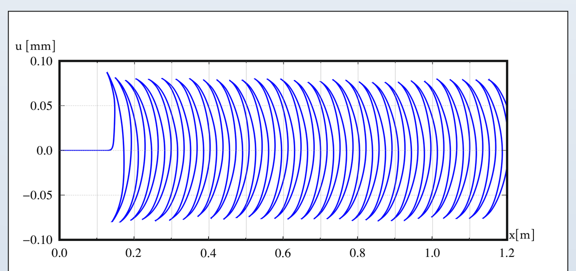

Figure 6: Displacement of the free end with a compressive load of 29 N and v_n=0.3 m/s.

FLUTTER IN TRACTION

The system described in Figure 3 is composed by two rigid rods (with mass per unit length ρ_1 and ρ_2) internally connected through a slider, while the entire system is constrained by an external hinge. The elasticity of the system is provided by the linear and rotational springs of stiffness k_1 and k_2. The defomed shape of the structure is fully determined by the two Lagrangian parameters θ and η. The follower load is induced to the column with a wheel sliding with Coulomb friction in the plane that moves with velocity v_n.

The two nonlinear differential equations of motion for the system, assuming that the spring are viscoelastic of the Voigt type with constants β_1, β_2, k_1 and k_2, is found with the virtual work principle

$$\begin{cases} \frac{1}{3}\rho_1 l_1^3(1+3\eta^2)\phi + \frac{1}{12}\rho_2 l_2^3\phi + \frac{1}{2}\rho_1 l_1^2\eta + 2\rho_1 l_1\dot{\phi}\dot{\eta}\eta + \beta_2\dot{\phi} + k_2\phi + \eta P\left(A_p^r\right) = 0 \\ \frac{1}{2}\rho_1 l_1^2\phi + \rho_1 l_1(\eta - \dot{\phi}^2\eta +)\beta_1\dot{\eta} + k_1\eta = 0 \end{cases}$$

As in the Ziegler's column, the approximation employed by Oden and Martins is used to simplify the frictional problem.

If we consider the absence of viscosity (β_1=β_2=0) the solutions of the linearized system can be written in the matrix form

$$\begin{pmatrix} \omega^2[4+1/(\rho\lambda^3)]-12/k & 6(\omega^2-2\gamma) \\ \omega^2 & 2(\omega^2-1) \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where A_j are the (complex) amplitudes, Ω is the circular frequency and the other parameters are

$$\gamma = \frac{P}{k_1 l_1} \qquad \omega^2 = \frac{\rho_1 l_1}{k_1}\Omega^2 \qquad \rho = \frac{\rho_1}{\rho_2} \qquad k = \frac{k_1 l_1^2}{k_2} \qquad \lambda = \frac{l_1}{l_2}$$

Non-trivial solution to the system is possible if the determinant of the matrix becomes zero. In that case three possibilities arise:

- two real positive values for ω^2, a situation that corresponds to **stability** of the system ($\gamma<\gamma_f$). It is shown in Figure 7;
- two complex conjugate values for ω^2, a situation where the oscillation of the system becomes steady at large deformation ($\gamma_f<\gamma<\gamma_d$). This instability is called **flutter**. It is shown in Figure 8;
- two real and negative values for ω^2, a situation corresponding to vibrations that grow exponentially with time ($\gamma>\gamma_d$). This instability is called **divergence**.

The critical load for flutter and divergence instability is written in the adimensional form

$$\gamma_{f,d} = \frac{4\rho\lambda^3(k+3)+k \mp 4\sqrt{3k\rho\lambda^3(1+\rho\lambda^3))}}{6\rho\lambda^3 k}$$

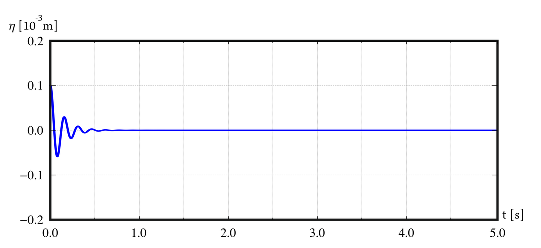

Figure 7: Parameter η with a tensile load P=5 N and v_n=0.3 m/s. The system results stable.

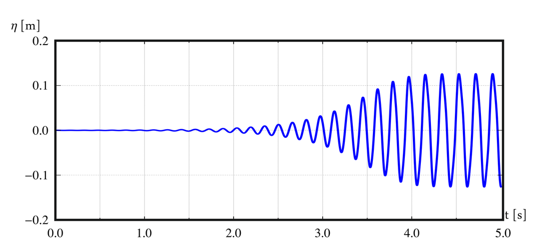

Figure 8: Parameter η with a tensile load P=12 N and v_n=0.3 m/s. The system shows flutter instability.

LOAD FRAME

The testing machine that performs the experiments is realized with a conveyor belt (Figure 9), on which are arranged the elastic structures object of study. The control pannel allows adjustment of the velocity of the plane and inversion of the motion direction. This permits to perform experiments with compressive load and also tensile load. The weight is applied to the system with the aid of a loading floor, free to slide in four linear bushing. The structures previously exposed (Figure 10, 11, 12) are mounted on a frame directly fixed to the conveyor belt.

Figure 10: Testing machine used in the experiments.

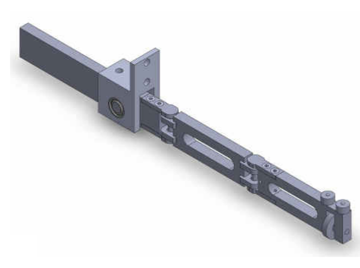

Figure10: Ziegler's column realized for the experiments.

Figure 11: Beck's column realized for the experiments.

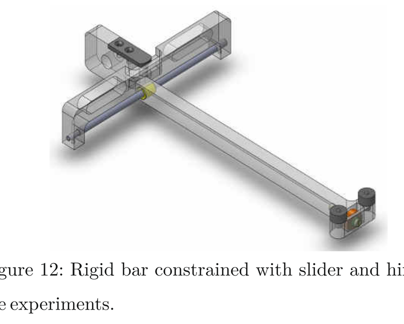

Figure 12: Rigid bar constrained with slider and hinge for the experiments.

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