PRECESSION ON A ROTATING SADDLE: A GYRO FORCE IN AN INERTIAL FRAME

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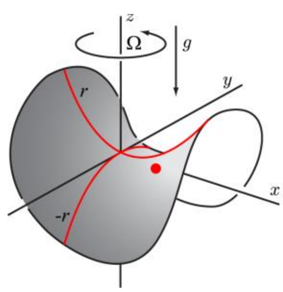
Summary Particles in rotating saddle potentials exhibit precessional motion which, up to now, has been explained by explicit computation. We show that this precession is due to a hidden gyroscopic force which, unlike the standard Coriolis force, is present in the inertial frame. We do so by finding a hodograph-like "guiding center" transformation using the method of normal form, which yields a simplified equation for the guiding center of the trajectory that coincides with the equation of the Foucault's pendulum. In this sense, a particle trapped in the symmetric rotating saddle trap is, effectively, a Foucault's pendulum, but in the inertial frame.

INTRODUCTION

The existence of Trojan asteroids in a triangular Lagrange libration point on the orbit of Jupiter is a consequence of the basic fact that a particle can be trapped in the rotating saddle potential [4,9]. Stability conditions for a heavy particle sliding without friction on a rotating saddle surface in the presence of gravity (Fig. 1) were obtained by Brouwer as early as 1918 [1,11]. Brouwer explicitly demonstrated that the saddle can be stabilized by rotation of the potential (in two dimensions) in contrast to the Paul trap for suspending charged particles in an oscillating electric field by analogy with the so-called Stephenson-Kapitsa pendulum in which the upside-down equilibrium is stabilized by vibration of the pivot [3]. Stabilization of the unstable saddle equilibrium by both rotation and vibration is well described by the concept of effective potential, which is equal to the kinetic energy of the high frequency motions of the system generated by the imposed oscillating field and affects the energy spectrum of the averaged motion [5-7]. In the case when the saddle potential is symmetric, the trajectory of the particle trapped by the rotating saddle in the non-rotating frame exhibits a slow prograde precession, Fig 1. This somewhat mysterious precession discovered first in the context of accelerator physics [2] and particle traps [8-10] is specific in that the standard averaging methods do not grasp the phenomenon [6].



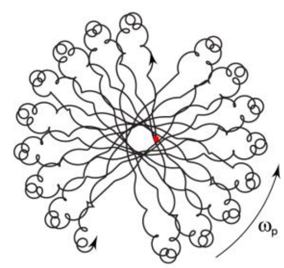


Fig. 1 (Left) A heavy particle on a symmetric rotating saddle surface and (right) prograde precession of its orbit in the non-rotating frame (figure adapted from [12])

HODOGRAPH TRANSFORMATION AND THE GUIDING CENTER EQUATION

We consider a point mass sliding without friction on a saddle surface rotating about a vertical axis with the angular velocity $\omega = \varepsilon^{-1}$, $\varepsilon \ll 1$, Fig 1. Assuming the principal curvatures of the saddle at the equilibrium point to be equal, linearized equations near the equilibrium in the non-rotating frame after appropriate rescaling of time take form

$$\ddot{x} + S(\omega t)x = 0,\ x \in \mathbb{R}^2, \tag{1}$$

where

$$S(\tau) = \begin{pmatrix} \cos 2\tau & \sin 2\tau \\ \sin 2\tau & -\cos 2\tau \end{pmatrix}, \quad \tau = \omega t.$$

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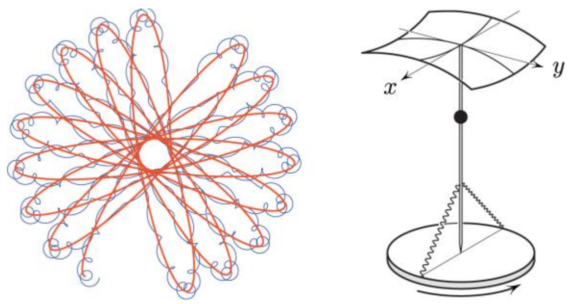

Fig. 2 (Left) Trajectory of the guiding center *u* (thick curve) tracking the corresponding trajectory *x* (thin curve). The view is in the inertial frame with ε= 0.45. (Right) A possible mechanical realization of the rotating saddle trap. Here, *x* and *y* are the angular variables of the inverted pendulum, and the graph of the potential energy is shown (figure adapted from [12]).

Theorem. Given a vector function $x:\mathbb{R}\to\mathbb{R}^2$, consider its "guiding center", or the "hodograph" image

$$u = x - \tfrac{\varepsilon^2}{4}S(t/\varepsilon)(x-\varepsilon J\dot{x}), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $x(t)$ is a solution of (1), then $u(t)$ satisfies

$$\ddot{u} - \tfrac{\varepsilon^3}{4}J\dot{u} + \tfrac{\varepsilon^2}{4}u = \varepsilon^4 f(u,\dot{u},\varepsilon),$$

where f is a function linear in u, $\dot{u}$ and analytic in ε, in a fixed neighbourhood of $\varepsilon=0$. The guiding center therefore behaves, (ignoring the $O(\varepsilon^4)$ - terms) as a point charge of unit mass in the potential $V(u)=\frac{\varepsilon^2}{8}u^2$ in the magnetic field of constant magnitude $B=\varepsilon^3/4$ perpendicular to the u-plane.

Fig.2 illustrates a possible mechanical implementation of the rotating saddle trap as a light rod with a massive ball mounted on a turntable via a ball joint. Two springs are attached to the rod, and the height of the ball is adjustable, like in a metronome. If the ball is placed sufficiently low then the springs will stabilize the pendulum in the *x*-direction while the *y*-direction remains unstable; thus, the potential acquires a saddle shape.

CONCLUSIONS

We demonstrated that the rapid rotation of the symmetric saddle potential creates a weak Lorentz-like, or a Coriolis-like force, in addition to an effective stabilizing potential - all in the inertial frame. As a result, the particle in the rotating saddle exhibits, in addition to oscillations caused by effective restoring force, a slow prograde precession in the inertial frame caused by this pseudo-Coriolis effect. By finding a hodograph-like "guiding center" transformation using the method of normal form, we found the effective equations of this precession that coincide with the equations of the Foucault's pendulum.

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References

[1] Brouwer L.E. Beweging van een materieel punt op den bodem eener draaiende vaas onder den invloed der zwaartekracht. *N. Arch. v. Wisk.* **2**: 407–419, 1918.
[2] Pearce R. M. Strong focussing in a magnetic helical quadrupole channel. *Nucl. Instrum. Methods* **83**(1): 101–108, 1970.
[3] Paul W. Electromagnetic traps for charged and neutral particles, *Rev. Mod. Phys.* **62**: 531–540, 1990.
[4] Bialynicki-Birula I., Kalinski M., Eberly J. H. Lagrange equilibrium points in celestial mechanics and nonspreading wave packets for strongly driven Rydberg electrons, *Phys. Rev. Lett.* **73**:1777–1780, 1994.
[5] Veselic K. On the stability of rotating systems, *Z. Angew. Math. Mech.* **75**: 325–328, 1995.
[6] Shapiro V.E. The gyro force of high-frequency fields lost by the concept of effective potential. *Phys. Lett. A* **238**: 147–152, 1998.
[7] Levi M., Geometry of Kapitsa's potentials. *Nonlinearity* **11**: 1365–1368, 1998.
[8] Thompson R.., Harmon T., Ball M. The rotating-saddle trap: A mechanical analogy to RF-electric-quadrupole ion trapping? *Can. J. Phys.* **80**: 1433–48, 2002.
[9] Bialynicki-Birula I., Bialynicka-Birula Z., Chmura B. Trojan states of electrons guided by Bessel beams. *Laser Physics* **15**(10): 1371–1380, 2005.
[10] Hasegawa T., Bollinger J. J. Rotating-radio-frequency ion traps. *Phys. Rev. A* **72**: 043403, 2005.
[11] Kirillov O. N. *Nonconservative Stability Problems of Modern Physics*, De Gruyter, Berlin/Boston, 2013.
[12] Kirillov O. N., Levi M. Rotating saddle trap as Foucault's pendulum. *Am. J. Phys.* **84**(1): 26–31, 2016.